

# Categorification of Algebras: 2-Algebras

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## Abstract

This paper introduces a categorification of  $k$ -algebras called 2-algebras, where  $k$  is a commutative ring. We define the 2-algebras as a 2-category with single object in which collections of all 1-morphisms and all 2-morphisms are  $k$ -algebras. It is shown that the category of 2-algebras is equivalent to the category of crossed modules in commutative  $k$ -algebras. Also we define the notion of homotopy for 2-algebras and we explore the relations of crossed module homotopy and 2-algebra homotopy.

## Introduction

The term “categorification” coined by Louis Crane refers to the process of replacing set theoretic concepts by category-theoretic analogues in mathematics. A categorified version of a group is a 2-group. Internal categories in the category of groups are exactly the same as 2-groups. The Brown-Spencer theorem [6] thus constructs the associated 2-group of a crossed module given by Whitehead [22] to define an algebraic model for a “(connected) homotopy 2-type”. The fact that the composition in the internal category must be a group homomorphism implies that the “interchange law” must hold. This equation is in fact equivalent via the Brown-Spencer result to the Peiffer identity.

We will be concerned in this paper exclusively with categorification of algebras. We will obtain analogous results in (commutative) algebras with regard to Porter’s work [18]. He states that there is an equivalence of categories between the category of internal categories in the category of  $k$ -algebras and the category of crossed modules of commutative  $k$ -algebras. Since the internal category in the category of  $k$ -algebras is a categorification of  $k$ -algebras, this internal category will be called as “strict 2-algebra” in this work. We define the strict 2-algebra by means of 2-module being a category in the category of modules as a 2-category with single object in which collections of 1-morphisms and 2-morphisms are  $k$ -algebras and we

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<sup>1</sup>2010 Mathematics Subject Classification: 18F99, 18G30, 18G55, 13C60  
Key words: 2-categories, Crossed modules, Homotopy.

denote the category of strict 2-algebras by **2Alg**. Given a group  $G$ , it is known that automorphisms of  $G$  yield a 2-group. Analogous result in algebras can be given that multiplications of  $C$  yield a strict 2-algebra where  $C$  is an  $R$ -algebra and  $R$  is a  $k$ -algebra.

A crossed module  $\mathcal{A} = (\partial : C \longrightarrow R)$  of commutative algebras is given by an algebra morphism  $\partial : C \longrightarrow R$  together with an action  $\cdot$  of  $R$  on  $C$  such that the relations below hold for each  $r \in R$  and each  $c, c' \in C$ ,

$$\begin{aligned}\partial(r \cdot c) &= r\partial(c) \\ \partial(c) \cdot c' &= cc' .\end{aligned}$$

Group crossed modules were firstly introduced by Whitehead in [20],[21]. They are algebraic models for homotopy 2-types, in the sense that [4],[14] the homotopy category of the model category [5],[8] of group crossed modules is equivalent to the homotopy category of the model category [10] of pointed 2-types: pointed connected spaces whose homotopy groups  $\pi_i$  vanish, if  $i \geq 3$ . The homotopy relation between crossed module maps  $\mathcal{A} \longrightarrow \mathcal{A}'$  was given by Whitehead in [21], in the context of “homotopy systems” called free crossed complexes.

In [1] it is addressed the homotopy theory of maps between crossed modules of commutative algebras. It is proven that if  $\mathcal{A}$  and  $\mathcal{A}'$  are crossed modules of algebras without any restriction on  $\mathcal{A}$  and  $\mathcal{A}'$  then the crossed module maps  $\mathcal{A} \longrightarrow \mathcal{A}'$  and their homotopies give a groupoid.

In this paper we show that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras. In [13], it is given an equivalence between the category of crossed modules in associative algebras and the category of strict associative 2-algebras defined by means of 2-vector space. Also we define the notion of homotopy for 2-algebras. This definition is essentially a special case of 2-natural transformation due to Gray in [11]. And we explore the relations between the crossed module homotopies and 2-algebra homotopies. Similar results are given [12] by İçen for 2-groupoids.

## 1 Internal Categories and 2-categories

We begin by recalling internal categories as well as 2-categories. Ehresmann defined internal categories in [9], and by now they are an important part of category theory [7].

### 1.1 Internal categories

**Definition 1.1** *Let  $\mathbf{C}$  be any category. An internal category in  $\mathbf{C}$ , say  $\mathbf{A}$ , consists of:*

- ◆ *an object of objects  $A_0 \in \mathbf{C}$*

- ◆ an object of morphisms  $A_1 \in \mathbf{C}$ ,
- together with
- ◆ source and target morphisms  $s, t : A_1 \longrightarrow A_0$ ,
- ◆ an identity-assigning morphism  $e : A_0 \longrightarrow A_1$ ,
- ◆ a composition morphism  $\circ : A_1 \times_{A_0} A_1 \longrightarrow A_1$  such that the following diagrams commute, expressing the usual category laws:
- ◆ laws specifying the source and target of identity morphisms:

$$\begin{array}{ccc}
 A_0 & \xrightarrow{e} & A_1 \\
 & \searrow 1_{A_0} & \downarrow s \\
 & & A_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_0 & \xrightarrow{e} & A_1 \\
 & \searrow 1_{A_0} & \downarrow t \\
 & & A_0
 \end{array}$$

- ◆ laws specifying the source and target of composite morphisms:

$$\begin{array}{ccc}
 A_1 \times_{A_0} A_1 & \xrightarrow{\circ} & A_1 \\
 \rho_1 \downarrow & & \downarrow s \\
 A_1 & \xrightarrow{s} & A_0
 \end{array}$$
  

$$\begin{array}{ccc}
 A_1 \times_{A_0} A_1 & \xrightarrow{\circ} & A_1 \\
 \rho_2 \downarrow & & \downarrow t \\
 A_1 & \xrightarrow{t} & A_0
 \end{array}$$

- ◆ the associative law for composition of morphisms:

$$\begin{array}{ccc}
 A_1 \times_{A_0} A_1 \times_{A_0} A_1 & \xrightarrow{\circ} & A_1 \times_{A_0} A_1 \\
 \rho_2 \downarrow & & \downarrow t \\
 A_1 \times_{A_0} A_1 & \xrightarrow{t} & A_0
 \end{array}$$

- ◆ the left and right unit laws for composition of morphisms:

$$\begin{array}{ccccc}
 A_0 \times_{A_0} A_1 & \xrightarrow{e \times_{A_0} 1_{A_1}} & A_1 \times_{A_0} A_1 & \xleftarrow{1_{A_1} \times_{A_0} e} & A_1 \times_{A_0} A_0 \\
 & \searrow \rho_2 & \downarrow \circ & \swarrow \rho_1 & \\
 & & A_1 & & 
 \end{array}$$

The pullback  $A_1 \times_{A_0} A_1$  is defined via the square:

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 & \xrightarrow{\rho_2} & A_1 \\ \rho_1 \downarrow & & \downarrow s \\ A_1 & \xrightarrow{t} & A_0. \end{array}$$

We denote this internal category with  $A = (A_0, A_1, s, t, e, \circ)$ .

**Definition 1.2** Let  $\mathbf{C}$  be a category. Given internal categories  $A$  and  $A'$  in  $\mathbf{C}$ , an **internal functor** between them, say  $F : A \longrightarrow A'$ , consists of

◆ a morphism  $F_0 : A_0 \longrightarrow A'_0$ ,

◆ a morphism  $F_1 : A_1 \longrightarrow A'_1$

such that the following diagrams commute, corresponding to the usual laws satisfied by a functor:

◆ preservation of source and target:

$$\begin{array}{ccc} A_1 & \xrightarrow{s} & A_0 \\ F_1 \downarrow & & \downarrow F_0 \\ A'_1 & \xrightarrow{s'} & A'_0 \end{array} \quad \begin{array}{ccc} A_1 & \xrightarrow{t} & A_0 \\ F_1 \downarrow & & \downarrow F_0 \\ A'_1 & \xrightarrow{t'} & A'_0 \end{array}$$

◆ preservation of identity morphisms:

$$\begin{array}{ccc} A_0 & \xrightarrow{e} & A_1 \\ F_0 \downarrow & & \downarrow F_1 \\ A'_0 & \xrightarrow{e'} & A'_1 \end{array}$$

◆ preservation of composite morphisms:

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 & \xrightarrow{F_1 \times_{A_0} F_1} & A'_1 \times_{A'_0} A'_1 \\ \circ \downarrow & & \downarrow \circ' \\ A_1 & \xrightarrow{F_1} & A'_1 \end{array}$$

Given two internal functors  $F : A \longrightarrow A'$  and  $G : A' \longrightarrow A''$  in some category  $\mathbf{C}$ , we define their composite  $FG : A \longrightarrow A''$  by taking  $(FG)_0 = F_0 G_0$  and  $(FG)_1 = F_1 G_1$ . Similarly, we define the identity internal functor in  $\mathbf{C}$ ,  $1_A : A \longrightarrow A$  by taking  $(1_A)_0 = 1_{A_0}$  and  $(1_A)_1 = 1_{A_1}$ .

**Definition 1.3** Let  $\mathbf{C}$  be a category. Given two internal functors  $F, G : A \longrightarrow A'$  in  $\mathbf{C}$ , an **internal natural transformation** in  $\mathbf{C}$  between them, say  $\theta : F \Longrightarrow G$ , is a morphism  $\theta : A_0 \longrightarrow A'_1$  for which the following diagrams commute, expressing the usual laws satisfied by a natural transformation:

◆ laws specifying the source and target of a natural transformation:

$$\begin{array}{ccc} A_0 & \xrightarrow{\theta} & A'_1 \\ & \searrow F_0 & \downarrow s' \\ & & A'_0 \end{array} \quad \begin{array}{ccc} A'_0 & \xrightarrow{\theta} & A'_1 \\ & \searrow G_0 & \downarrow t' \\ & & A_0 \end{array}$$

◆ the commutative square law:

$$\begin{array}{ccc} A_1 & \xrightarrow{\Delta(s\theta \times G)} & A'_1 \times_{A'_0} A'_1 \\ \downarrow \Delta(F \times t\theta) & & \downarrow \circ' \\ A'_1 \times_{A'_0} A'_1 & \xrightarrow{\circ'} & A'_1 \end{array}$$

Given an internal functor  $F : A \longrightarrow A'$  in  $\mathbf{C}$ , the identity internal natural transformation  $1_F : F \Longrightarrow F$  in  $\mathbf{C}$  is given by  $1_F = F_0 e$ .

## 1.2 2-categories

**Definition 1** A 2-category  $\mathcal{G}$  consists of a class of objects  $G_0$  and for any pair of objects  $(A, B)$  a small category of morphisms  $\mathcal{G}(A, B)$ -with objects  $G_1(A, B)$  and morphisms  $G_2(A, B)$ -, along with composition functors

$$\bullet : \mathcal{G}(A, B) \times \mathcal{G}(B, C) \longrightarrow \mathcal{G}(A, C)$$

for every triple  $(A, B, C)$  of objects and identity functors from the terminal category to  $\mathcal{G}(A, A)$

$$i_A : 1 \longrightarrow \mathcal{G}(A, A)$$

for all objects  $A$  such that  $\bullet$  is associative and

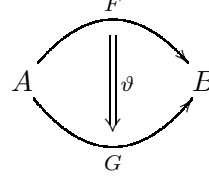
$$F \bullet i_B = F = i_A \bullet F \quad \text{as well as} \quad \vartheta \bullet I_{i_B} = \vartheta = I_{i_A} \bullet \vartheta$$

hold for all  $F \in G_1(A, B)$  and  $\vartheta \in G_2(A, B)$  where source and target morphisms are defined by

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ s : G_1(A, B) & \longrightarrow & G_0 \\ F & \longmapsto & s(F) = A \end{array}$$

$$\begin{array}{ccc} t : G_1(A, B) & \longrightarrow & G_0 \\ F & \longmapsto & t(F) = B \end{array}$$

for  $F \in G_1(A, B)$  and



$$\begin{array}{ccc} s : G_2(A, B) & \longrightarrow & G_1 \\ \vartheta & \longmapsto & s(\vartheta) = F \end{array}$$

$$\begin{array}{ccc} t : G_2(A, B) & \longrightarrow & G_0 \\ \vartheta & \longmapsto & t(\vartheta) = G \end{array}$$

for  $\vartheta : F \longrightarrow G \in G_2(A, B)$ . For all pairs of objects  $(A, B)$  elements of  $G_1(A, B)$  are called 1-morphisms or 1-cells of  $\mathcal{G}$  and elements of  $G_2(A, B)$  are called 2-morphisms or 2-cells of  $\mathcal{G}$ . We write  $G_1$  and  $G_2$  for the classes of all 1-morphisms and 2-morphisms respectively.

There are two ways of composing 2-morphisms: using the composition  $\circ$  inside the categories  $\mathcal{G}(A, B)$ , called vertical composition, and using the morphism level of the functor  $\bullet$ , called horizontal composition. These compositions must satisfy the following equation: for  $\alpha, \alpha' \in G_2(A, B)$  with  $t(\alpha) = s(\alpha')$  and  $\gamma, \gamma' \in G_2(B, C)$  with  $t(\gamma) = s(\gamma')$

$$(\alpha \circ \alpha') \bullet (\gamma \circ \gamma') = (\alpha \bullet \gamma) \circ (\alpha' \bullet \gamma')$$

which is called “interchange law”.

## 2 Constructions of Two-Algebras

In this section we will construct 2-algebras by categorification. We can categorify the notion of an algebra by replacing the equational laws by isomorphisms satisfying extra structure and properties we expect. In [3] Baez and Crans introduce the Lie 2-algebra by means of the concept of 2-vector space defined as an internal category in the category of vector spaces by them. Obviously we get a new notion of “2-module” which can be considered as an internal category in the category of modules and we categorify the notion of an algebra.

## 2.1 2-Modules

A categorified module or “2-module” should be a category with structure analogous to that of a  $k$ -module, with functors replacing the usual  $k$ -module operations. Here we instead define a 2-module to be an internal category in a category of  $k$ -modules  $\mathbf{Mod}$ . Since the main component part of a  $k$ -algebra is a  $k$ -module, a 2-algebra will have an underlying 2-module of this sort. In this section we thus first define a category of these 2-modules.

In the rest of this paper, the terms a module and an algebra will always refer to a  $k$ -module and a  $k$ -algebra.

**Definition 2.1** *A 2-module is an internal category in  $\mathbf{Mod}$ .*

Thus, a 2-module  $M$  is a category with a module of objects  $M_0$  and a module of morphisms  $M_1$ , such that the source and target maps  $s, t : M_1 \rightarrow M_0$ , the identity assigning map  $e : M_0 \rightarrow M_1$ , and the composition map  $\circ : M_1 \times_{M_0} M_1 \rightarrow M_1$  are all module morphisms. We write a morphism as  $a : x \rightarrow y$  when  $s(a) = x$  and  $t(a) = y$ , and sometimes we write  $e(x)$  as  $1_x$ .

The following proposition is given for the  $\mathbf{Vect}$  vector space category in [3]. But we rewrite this proposition for  $\mathbf{Mod}$ .

**Proposition 2.2** *It is defined a 2-module by specifying the modules  $M_0$  and  $M_1$  along with the source, target and identity module morphisms and the composition morphism  $\circ$ , satisfying the conditions of Definition 1.1. The composition map is uniquely determined by*

$$\begin{aligned} \circ : M_1 \times_{M_0} M_1 &\longrightarrow M_1 \\ (a, b) &\longmapsto \circ(a, b) = a \circ b = a + b - (es)(b). \end{aligned}$$

**Proof:** First given modules  $M_0$ ,  $M_1$  and module morphisms  $s, t : M_1 \rightarrow M_0$  and  $e : M_0 \rightarrow M_1$ , we will define a composition operation that satisfies the laws in the definition of internal category, obtaining a 2-module.

Given  $a, b \in M_1$  such that  $t(a) = s(b)$ , i.e.

$$a : x \rightarrow y \text{ and } b : y \rightarrow z$$

we define their composite  $\circ$  by

$$\begin{aligned} \circ : M_1 \times_{M_0} M_1 &\longrightarrow M_1 \\ (a, b) &\longmapsto \circ(a, b) = a \circ b = a + b - (es)(b). \end{aligned}$$

We will show that with this composition  $\circ$  the diagrams of the definition of internal category commute. The triangles specifying the source and target of the identity-assigning morphism do not involve composition. The second pair of diagrams commute since

$$\begin{aligned} s(a \circ b) &= s(a + b - (es)(b)) \\ &= s(a) + s(b) - (se)(s(b)) \\ &= s(a) + s(b) - s(b) \\ &= s(a) = x \end{aligned}$$

and since  $t(a) = s(b)$ ,

$$\begin{aligned}
t(a \circ b) &= t(a + b - (es)(b)) \\
&= t(a) + t(b) - (te)(s(b)) \\
&= t(a) + t(b) - s(b) \\
&= t(b) = z.
\end{aligned}$$

Since module operation is associative, the associative law holds for composition. The left and right unit laws are satisfied since given  $a : x \longrightarrow y$ ,

$$\begin{aligned}
e(x) \circ a &= e(x) + a - (es)(a) \\
&= e(x) + a - e(x) \\
&= a
\end{aligned}$$

and

$$\begin{aligned}
a \circ e(y) &= a + e(y) - (es)(e(y)) \\
&= a + e(y) - e(y) \\
&= a.
\end{aligned}$$

We thus have a 2-module.

Given a 2-module  $M$ , we show that its composition must be defined by the formula given above. Let  $(a, g)$  and  $(a', g')$  be composable pairs of morphisms in  $M_1$ , i.e.

$$a : x \longrightarrow y \text{ and } b : y \longrightarrow z$$

and

$$a' : x' \longrightarrow y' \text{ and } b' : y' \longrightarrow z'.$$

Since the source and target maps are module morphisms,  $(a + a', b + b')$  also forms a composable pair, and since that the composition is module morphism

$$(a + a') \circ (b + b') = a \circ b + a' \circ b'.$$

Then if  $(a, b)$  is a composable pair, i.e,  $t(a) = s(b)$ , we have

$$\begin{aligned}
a \circ b &= (a + 1_{M_1}) \circ (1_{M_1} + b) \\
&= (a + e(s(b) - s(b))) \circ (e(s(b) - s(b)) + b) \\
&= (a - e(s(b)) + e(s(b))) \circ (e(s(b)) - e(s(b)) + b) \\
&= (a \circ e(s(b))) + (-e(s(b)) + e(s(b))) \circ (-e(s(b)) + b) \\
&= a \circ e(s(b)) + (-e(s(b)) \circ (-e(s(b)))) + (e(s(b)) \circ b) \\
&= a - e(s(b)) + b \\
&= a + b - e(s(b)).
\end{aligned}$$

This show that we can define  $\circ$  by

$$\begin{aligned}
\circ : M_1 \times_{M_0} M_1 &\longrightarrow M_1 \\
(a, b) &\longmapsto \circ(a, b) = a \circ b = a + b - e(s(b)).
\end{aligned}$$

□



**Corollary 2.3** For  $b \in \ker s$ , we have

$$\begin{aligned} a \circ b &= a + b - (es)(b) \\ &= a + b. \end{aligned}$$

**Definition 2.4** Let  $M$  and  $N$  be 2-modules, a 2-module functor  $F : M \longrightarrow N$  is an internal functor in **Mod** from  $M$  to  $N$ . 2-modules and 2-module functors between them is called the category of 2-modules denoted by **2Mod**.

After we get the definition of a 2-module, we define the definition of a categorified algebra which is main concept of this paper.

## 2.2 Two-algebras

**Definition 2.5** A weak 2-algebra consists of

◆ a 2-module  $A$  equipped with a functor  $\bullet : A \times A \longrightarrow A$ , which is defined by  $(x, y) \mapsto x \bullet y$  and bilinear on objects and defined by  $(f, g) \mapsto f \bullet g$  on morphisms satisfying interchange law, i.e.,

$$(f_1 \bullet g_1) \circ (f_2 \bullet g_2) = (f_1 \circ f_2) \bullet (g_1 \circ g_2)$$

◆  $k$ -bilinear natural isomorphisms

$$\alpha_{x,y,z} : (x \bullet y) \bullet z \longrightarrow x \bullet (y \bullet z)$$

$$l_x : 1 \bullet x \longrightarrow x$$

$$r_x : x \bullet 1 \longrightarrow x$$

such that the following diagrams commute for all objects  $w, x, y, z \in A_0$ .

$$\begin{array}{ccc} ((w \bullet x) \bullet y) \bullet z & \xrightarrow{\alpha_{w \bullet x, y, z}} & (w \bullet x) \bullet (y \bullet z) \\ \alpha_{w, x, y} \bullet 1_z \downarrow & & \searrow \alpha_{w, x, y \bullet z} \\ (w \bullet (x \bullet y)) \bullet z & \xrightarrow{\alpha_{w, x \bullet y, z}} & w \bullet ((x \bullet y) \bullet z) \xrightarrow{1_w \bullet \alpha_{x, y, z}} w \bullet (x \bullet (y \bullet z)) \end{array}$$
  

$$\begin{array}{ccc} (x \bullet 1) \bullet y & \xrightarrow{\alpha_{x, 1, y}} & x \bullet (1 \bullet y) \\ \searrow r_x \bullet 1_y & & \downarrow 1_x \bullet l_y \\ & & x \bullet y \end{array}$$

A strict 2-algebra is the special case where  $\alpha_{x,y,z}$ ,  $l_x$ ,  $r_x$  are all identity morphisms. In this case we have

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

$$1 \bullet x = x, x \bullet 1 = x$$

Strict 2-algebra is called commutative strict 2-algebra if  $x \bullet y = y \bullet x$  for all objects  $x, y \in A_0$  and  $f \bullet g = g \bullet f$  for all morphisms  $f, g \in A_1$ .

In the rest of this paper, the term 2-algebra will always refer to a commutative strict 2-algebra. A homomorphism between 2-algebras should preserve both the 2-module structure and the  $\bullet$  functor.

**Definition 2.6** *Given 2-algebras  $A$  and  $A'$ , a homomorphism*

$$F : A \longrightarrow A'$$

*consists of*

- ◆ *a linear functor  $F$  from the underlying 2-module of  $A$  to that of  $A'$ ,*
- and*
- ◆ *a bilinear natural transformation*

$$F_2(x, y) : F_0(x) \bullet F_0(y) \longrightarrow F_0(x \bullet y)$$

- ◆ *an isomorphism  $F : 1' \longrightarrow F_0(1)$  where  $1$  is the identity object of  $A$  and  $1'$  is the identity object of  $A'$ ,*
- such that the following diagrams commute for  $x, y, z \in A_0$ ,*

$$\begin{array}{ccccc} (F(x) \bullet F(y)) \bullet F(z) & \xrightarrow{F_2 \bullet 1} & F(x \bullet y) \bullet F(z) & \xrightarrow{F_2} & F((x \bullet y) \bullet z) \\ \downarrow \alpha_{F(x), F(y), F(z)} & & & & \downarrow F(\alpha_{x, y, z}) \\ F(x) \bullet (F(y) \bullet F(z)) & \xrightarrow{1 \bullet F_2} & F(x) \bullet F(y \bullet z) & \xrightarrow{F_2} & F(x \bullet (y \bullet z)). \end{array}$$

$$\begin{array}{ccc} 1' \bullet F(x) & \xrightarrow{l'_{F(x)}} & F(x) \\ F_0 \bullet 1 \downarrow & & \uparrow F(l_x) \\ F(1) \bullet F(x) & \xrightarrow{F_2} & F(1 \bullet x). \end{array}$$

$$\begin{array}{ccc} F(x) \bullet 1' & \xrightarrow{r'_{F(x)}} & F(x) \\ 1 \bullet F_0 \downarrow & & \uparrow F(r_x) \\ F(x) \bullet F(1) & \xrightarrow{F_2} & F(x \bullet 1). \end{array}$$

**Definition 2.7** *2-algebras and homomorphisms between them give the category of 2-algebras denoted by  $2\mathbf{Alg}$ .*

Therefore if  $A = (A_0, A_1, s, t, e, \circ, \bullet)$  is a 2-algebra,  $A_0$  and  $A_1$  are algebras with this  $\bullet$  bilinear functor. Thus we can take that 2-algebra is a 2-category with a single object say  $*$ , and  $A_0$  collections of its 1-morphisms and  $A_1$  collections of its 2-morphisms are algebras with identity.

### 2.3 Multiplication Algebras yield a 2-algebra

In [17] Norrie developed Lue's work, [15] and introduced the notion of an actor of crossed modules of groups where it is shown to be the analogue of the automorphism group of a group. In the category of commutative algebras the appropriate replacement for automorphism groups is the multiplication algebra  $\mathcal{M}(C)$  of an algebra  $C$  which is defined by MacLane [16].

Let  $C$  be an associative (not necessarily unitary or commutative)  $R$ -algebra. We recall Mac Lane's construction of the  $R$ -algebra  $\text{Bim}(C)$  of bimultipliers of  $C$  [16].

An element of  $\text{Bim}(C)$  is a pair  $(\gamma, \delta)$  of  $R$ -linear mappings from  $C$  to  $C$  such that

$$\begin{aligned}\gamma(cc') &= \gamma(c)c' \\ \delta(cc') &= c\delta(c')\end{aligned}$$

and

$$c\gamma(c') = \delta(c)c'.$$

$\text{Bim}(C)$  has an obvious  $R$ -module structure and a product

$$(\gamma, \delta)(\gamma', \delta') = (\gamma\gamma', \delta'\delta),$$

the value of which is still in  $\text{Bim}(C)$ .

Suppose that  $\text{Ann}(C) = 0$  or  $C^2 = C$ . Then  $\text{Bim}(C)$  acts on  $C$  by

$$\begin{aligned}\text{Bim}(C) \times C &\rightarrow C; & ((\gamma, \delta), c) &\mapsto \gamma(c), \\ C \times \text{Bim}(C) &\rightarrow C; & (c, (\gamma, \delta)) &\mapsto \delta(c)\end{aligned}$$

and there is a

$$\begin{aligned}\mu : C &\longrightarrow \text{Bim}(C) \\ c &\longmapsto (\gamma_c, \delta_c)\end{aligned}$$

with

$$\gamma_c(x) = cx \quad \text{and} \quad \delta_c(x) = xc.$$

*Commutative case:* we still assume  $\text{Ann}(C) = 0$  or  $C^2 = C$ . If  $C$  is a commutative  $R$ -algebra and  $(\gamma, \delta) \in \text{Bim}(C)$ , then  $\gamma = \delta$ . This is because for every  $x \in C$ :

$$\begin{aligned}x\delta(c) &= \delta(c)x = c\gamma(x) = \gamma(x)c \\ &= \gamma(xc) = \gamma(cx) = \gamma(c)x = x\gamma(c).\end{aligned}$$

Thus  $\text{Bim}(C)$  may be identified with the  $R$ -algebra  $\mathcal{M}(C)$  of multipliers of  $C$ . Recall that a multiplier of  $C$  is a linear mapping  $\lambda : C \rightarrow C$  such that for all  $c, c' \in C$

$$\lambda(cc') = \lambda(c)c'.$$

Also  $\mathcal{M}(C)$  is commutative as

$$\lambda'\lambda(xc) = \lambda'(\lambda(x)c) = \lambda(x)\lambda'(c) = \lambda'(c)\lambda(x) = \lambda\lambda'(cx) = \lambda\lambda'(xc)$$

for any  $x \in C$ . Thus  $\mathcal{M}(C)$  is the set of all multipliers  $\lambda$  such that  $\lambda\gamma = \gamma\lambda$  for every multiplier  $\gamma$ .

In [19] Porter states that automorphisms of a group  $G$  yield a 2-group. The appropriate analogue of this result in algebra case can be given. We claim that multiplications of an  $R$ -algebra  $C$  give a 2-algebra which is called a multiplication 2-algebra.

Let  $k$  be a commutative ring,  $R$  be a  $k$ -algebra with identity and  $C$  be a commutative  $R$ -algebra with  $\text{Ann}(C) = 0$  or  $C^2 = C$ . Take  $A_0 = \mathcal{M}(C)$  and say 1-morphisms to the elements of  $A_0$ . We define the action of  $\mathcal{M}(C)$  on  $C$  as follows:

$$\begin{array}{ccc} \mathcal{M}(C) \times C & \longrightarrow & C \\ (f, x) & \longmapsto & f \blacktriangleright x = f(x). \end{array}$$

Using the action of  $\mathcal{M}(C)$  on  $C$ , we can form the semidirect product

$$C \rtimes \mathcal{M}(C) = \{(x, f) | x \in C, f \in \mathcal{M}(C)\}$$

with multiplication

$$(x, f)(x', f') = (f \blacktriangleright x' + f' \blacktriangleright x + x'x, f'f).$$

Take  $A_1 = C \rtimes \mathcal{M}(C)$  and say 2-morphisms to the elements of  $A_1$ . Therefore we get the following diagram for  $(x, f) \in C \rtimes \mathcal{M}(C)$ ,

$$\begin{array}{ccc} & f & \\ & \curvearrowright & \\ C & \begin{array}{c} \parallel \\ (x, f) \\ \parallel \end{array} & C \\ & \curvearrowleft & \\ & g & \end{array}$$

and we define the source, target and identity assigning maps as follows;

$$\begin{array}{ccc} s : C \rtimes \mathcal{M}(C) & \longrightarrow & \mathcal{M}(C) \\ (x, f) & \longmapsto & s(x, f) = f \end{array} \quad \begin{array}{ccc} t : C \rtimes \mathcal{M}(C) & \longrightarrow & \mathcal{M}(C) \\ (x, f) & \longmapsto & t(x, f) = M_x \cdot f \end{array}$$

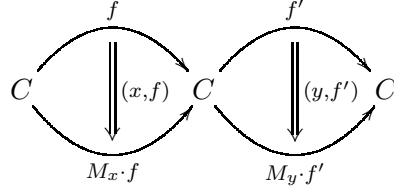
and

$$\begin{array}{ccc} e : \mathcal{M}(C) & \longrightarrow & C \rtimes \mathcal{M}(C) \\ f & \longmapsto & e(f) = (0, f) \end{array}$$

where  $M_x \cdot f$  is defined by  $(M_x \cdot f)(u) = xu + f(u)$  for  $u \in C$ .

There are two ways of composing 2-morphisms: vertical and horizontal composition. Now we define this compositions.

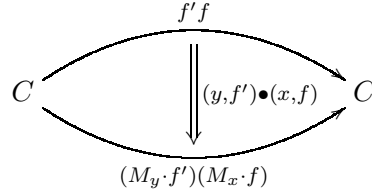
For  $(x, f), (y, f') \in C \rtimes \mathcal{M}(C)$



the horizontal composition is defined by

$$(x, f) \bullet (y, f') = (f'(x) + f(y) + xy, f'f),$$

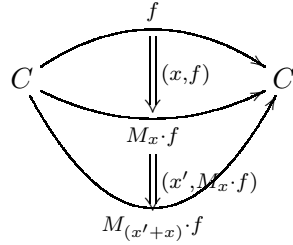
thus we have



and

$$\begin{aligned} t(f'(x) + f(y) + xy, f'f) &= M_{f'(x) + f(y) + xy} \cdot f'f \\ &= (M_y \cdot f')(M_x \cdot f) \end{aligned}$$

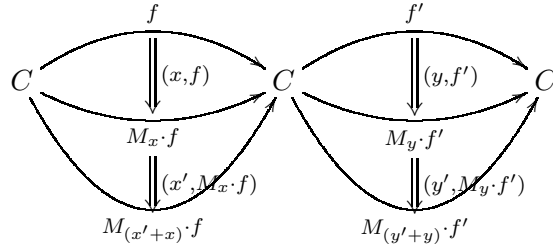
The vertical composition is defined by



$$(x, f) \circ (x', M_x \cdot f) = (x' + x, f)$$

for  $(x, f), (x', M_x \cdot f) \in C \rtimes \mathcal{M}(C)$  with  $t(x, f) = s(x', M_x \cdot f) = M_x \cdot f$ .

It remains to satisfy the interchange law, i.e.



$$[(x, f) \circ (x', M_x \cdot f)] \bullet [(y, f') \circ (y', M_y \cdot f')] = [(x, f) \bullet (y, f')] \circ [(x', M_x \cdot f) \bullet (y', M_y \cdot f')].$$

Evaluating the two sides separately, we get

$$\begin{aligned} \text{LHS} &= (x' + x, f) \bullet (y' + y, f') \\ &= (f'(x' + x) + f(y' + y) + (x' + x)(y' + y), f'f) \\ &= (f'(x') + f'(x) + f(y') + f(y) + x'y' + x'y + xy', f'f) \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= (f'(x) + f(y) + xy, f'f) \circ ((M_y \cdot f')(x') \\ &\quad + (M_x \cdot f)(y') + x'y', (M_y \cdot f')(M_x \cdot f)) \\ &= (f'(x) + f(y) + xy + (M_y \cdot f')(x') + (M_x \cdot f)(y') + x'y', f'f) \\ &= (f'(x) + f(y) + xy + yx' + f'(x') + xy' + f(y') + x'y', f'f) \end{aligned}$$

LHS and RHS are equal, thus interchange law is satisfied. Therefore we get a 2-algebra consists of the  $R$ -algebra  $C$  as single object and the  $R$ -algebra  $A_0$  of 1-morphisms and the  $R$ -algebra  $A_1$  of 2-morphisms.

### 3 Crossed modules and 2-algebras

Crossed modules have been used widely and in various contexts since their definition by Whitehead [22] in his investigations of the algebraic structure of relative homotopy groups. We recalled the definition of crossed modules of commutative algebras given by Porter [19].

Let  $R$  be a  $k$ -algebra with identity. A pre-crossed module of commutative algebras is an  $R$ -algebra  $C$  together with a commutative action of  $R$  on  $C$  and a morphism

$$\partial : C \longrightarrow R$$

such that for all  $c \in C, r \in R$

$$\text{CM1) } \partial(r \blacktriangleright c) = r\partial c.$$

This is a crossed  $R$ -module if in addition for all  $c, c' \in C$

$$\text{CM2) } \partial c \blacktriangleright c' = cc'.$$

The last condition is called the Peiffer identity. We denote such a crossed module by  $(C, R, \partial)$ .

A morphism of crossed modules from  $(C, R, \partial)$  to  $(C', R', \partial')$  is a pair of  $k$ -algebra morphisms  $\phi : C \longrightarrow C', \psi : R \longrightarrow R'$  such that

$$\partial' \phi = \psi \partial \quad \text{and} \quad \phi(r \blacktriangleright c) = \psi(r) \blacktriangleright \phi(c).$$

Thus we get a category  $\mathbf{XMod}_k$  of crossed modules (for fixed  $k$ ).

### Examples of Crossed Modules

1. Let  $I$  be an ideal in  $R$ . Then  $inc : I \rightarrow R$  is a crossed module. Conversely, if  $\partial : C \rightarrow R$  is a crossed module then the Peiffer identity implies that  $\partial C$  is an ideal in  $R$ .

2. Given any  $R$ -module  $M$ , the zero morphism  $0 : M \rightarrow R$  is a crossed module. Conversely: If  $(C, R, \partial)$  is a crossed module,  $\partial(C)$  acts trivially on  $\ker \partial$ , hence  $\ker \partial$  has a natural  $R/\partial(C)$ -module structure.

As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

3. Let  $\mathcal{M}(C)$  be multiplication algebra. Then  $(C, \mathcal{M}(C), \mu)$  is multiplication crossed module.  $\mu : C \rightarrow \mathcal{M}(C)$  is defined by  $\mu(r) = \delta_r$  with  $\delta_r(r') = rr'$  for all  $r, r' \in C$ , where  $\delta$  is multiplier  $\delta : C \rightarrow C$  such that for all  $r, r' \in C$ ,  $\delta(rr') = \delta(r)r'$ . Also  $\mathcal{M}(C)$  acts on  $C$  by  $\delta \blacktriangleright r = \delta(r)$ . (See [2] for details).

In [19] Porter states that there is an equivalence of categories between the category of internal categories in the category of  $k$ -algebras and the category of crossed modules of commutative  $k$ -algebras. In the following theorem, we will give a categorical presentation of this equivalence.

**Theorem 3.1** *The category of crossed modules  $\mathbf{XMod}_k$  is equivalent to that of 2-algebras,  $2Alg$ .*

**Proof:** Let  $A = (A_0, A_1, s, t, e, \circ, \bullet)$  be a 2-algebra consisting of a single object say  $*$  and an algebra  $A_0$  of 1-morphisms and an algebra  $A_1$  of 2-morphisms. For  $x, y \in A_0$  and  $f : x \rightarrow y \in A_1$ , we get the following diagram

$$\begin{array}{ccc} & x & \\ \curvearrowright & & \curvearrowleft \\ * & \begin{array}{c} \parallel \\ f \\ \parallel \end{array} & * \\ \curvearrowleft & & \curvearrowright \\ & y & \end{array}$$

We define  $s, t$  morphisms  $s : A_1 \rightarrow A_0, s(f) = x, t : A_1 \rightarrow A_0, t(f) = y$  and  $e$  morphism  $e : A_0 \rightarrow A_1$  for  $x \in A_0, e(x) : x \rightarrow x \in A_1$ .

The  $s, t$  and  $e$  morphisms are algebra morphisms and we have

$$\begin{aligned} se(x) &= s(e(x)) = x = Id_{A_0}(x) \\ te(x) &= t(e(x)) = x = Id_{A_0}(x) \end{aligned}$$

We define

$$\text{Ker } s = K = \{q \in A_1 \mid s(q) = Id_{A_0}\} \subseteq A_1$$

and  $\partial = t|_K$  algebra homomorphism by  $\partial : K \rightarrow A_0, \partial(q) = t(q)$ . We have semidirect product  $\text{Ker } s \rtimes A_0 = \{(q, x) \mid q \in \text{Ker } s, x \in A_0\}$  with multiplication  $(q, x) \bullet (q', x') = (x \blacktriangleright q' + x' \blacktriangleright q + q' \bullet q, x \bullet x')$  where action

of  $A_0$  on  $\text{Kers}$  is defined by  $x \blacktriangleright q = e(x) \bullet h$ . For each  $f \in A_1$ , we can write  $f = q + e(x)$  where  $q = f - es(f) \in \text{Kers}$  and  $x = s(f)$ . Suppose  $f' = q' + e(x')$ . Then

$$\begin{aligned} f \bullet f' &= (q + e(x)) \bullet (q' + e(x')) \\ &= q \bullet q' + q \bullet e(x') + e(x) \bullet q' + e(x) \bullet e(x') \\ &= e(x') \bullet q + e(x) \bullet q' + q \bullet q' + e(x \bullet x') \\ &= x' \blacktriangleright q + x \blacktriangleright q' + q \bullet q' + e(x \bullet x'). \end{aligned}$$

There is a map

$$\begin{aligned} \phi : A_1 &\longrightarrow \text{Kers} \rtimes A_0 \\ q + e(x) &\longmapsto \phi(q + e(x)) = (q, x). \end{aligned}$$

Now

$$\begin{aligned} \phi(f \bullet f') &= \phi(x' \blacktriangleright q + x \blacktriangleright q' + q \bullet q' + e(x \bullet x')) \\ &= (x' \blacktriangleright q + x \blacktriangleright q' + q \bullet q', x \bullet x') \\ &= (q, x) \bullet (q', x') \\ &= \phi(f) \bullet \phi(f') \end{aligned}$$

so  $\phi$  is a homomorphism. Also, there is an obvious inverse

$$\begin{aligned} \phi^{-1} : \text{Kers} \rtimes A_0 &\longrightarrow A_1 \\ (q, x) &\longmapsto \phi^{-1}(q, x) = q + e(x) \end{aligned}$$

which is also a homomorphism. Hence  $\phi$  is an isomorphism and we have established that  $\text{Ker } s \rtimes A_0 \simeq A_1$ . Since  $A$  is a 2-algebra and  $\text{Ker } s \rtimes A_0 \simeq A_1$ , we can define algebra morphisms

$$\begin{aligned} s : \text{Kers} \rtimes A_0 &\longrightarrow A_0 & t : \text{Kers} \rtimes A_0 &\longrightarrow A_0 \\ (q, x) &\longmapsto s(q, x) = x & (q, x) &\longmapsto t(q, x) = \partial(q) + x \end{aligned}$$

and

$$\begin{aligned} e : A_0 &\longrightarrow \text{Kers} \rtimes A_0 \\ x &\longmapsto e(x) = (0, x) \end{aligned}$$

and for  $t(q, x) = s(q', \partial(q) + x) = \partial(q) + x$  we define

$$\begin{aligned} \circ : \text{Kers} \rtimes A_0 \times_t \text{Kers} \rtimes A_0 &\longrightarrow \text{Kers} \rtimes A_0 \\ ((q, x), (q', \partial(q) + x)) &\longmapsto (q' + q, x) \end{aligned}$$

$$\begin{array}{c} \begin{array}{ccc} & x & \\ \curvearrowright & \Downarrow (q, x) & \curvearrowleft \\ * & & * \\ \curvearrowleft & \Downarrow \partial(q) + x & \curvearrowright \\ & \Downarrow (q', \partial(q) + x) & \\ & \partial(q' + q) + x & \end{array} & = & \begin{array}{ccc} & x & \\ \curvearrowright & \Downarrow (q' + q, x) & \curvearrowleft \\ * & & * \\ \curvearrowleft & \Downarrow \partial(q' + q) + x & \curvearrowright \end{array} \end{array}$$



which is vertical composition;

$$(q, x) \circ (q', \partial(q) + x) = (q' + q, x).$$

For  $(q, x), (p, y) \in \text{Kers} \rtimes A_0$ , horizontal composition is defined by

$$\begin{aligned} (q, x) \bullet (p, y) &= (x \blacktriangleright p + y \blacktriangleright q + p \bullet q, x \bullet y) \\ &= (e(x) \bullet p + e(y) \bullet q + p \bullet q, x \bullet y). \end{aligned}$$

Thus we have

CM1)

$$\begin{aligned} \partial(x \blacktriangleright q) &= \partial(e(x) \bullet q) \\ &= \partial(e(x)) \bullet \partial(q) \\ &= (te)(x) \bullet \partial(q) \\ &= x \bullet \partial(q). \end{aligned}$$

Also by interchange law we have

$$[(q, x) \bullet (p, y)] \circ [(q', \partial(q) + x) \bullet (p', \partial(p) + y)] = [(q, x) \circ (q', \partial(q) + x)] \bullet [(p, y) \circ (p', \partial(p) + y)].$$

Therefore, evaluating the two sides of this equation gives:

$$\begin{aligned} LHS &= (x \blacktriangleright p + y \blacktriangleright q + p \bullet q, x \bullet y) \\ &\quad \circ ((\partial(q) + x) \blacktriangleright p' + (\partial(p) + y) \blacktriangleright q' + p' \bullet q', (\partial(q) + x) \bullet (\partial(p) + y)) \\ &= ((\partial(q) + x) \blacktriangleright p' + (\partial(p) + y) \blacktriangleright q' + p' \bullet q' + x \blacktriangleright p + y \blacktriangleright q + p \bullet q, x \bullet y) \\ &= (\partial(q) \blacktriangleright p' + e(x) \bullet p' + \partial(p) \blacktriangleright q' \\ &\quad + e(y) \bullet q' + p' \bullet q' + e(x) \bullet p + e(y) \bullet q + p \bullet q, x \bullet y) \\ RHS &= (q' + q, x) \bullet (p' + p, y) \\ &= (x \blacktriangleright (p' + p) + y \blacktriangleright (q' + q) + (p' + p) \bullet (q' + q), x \bullet y) \\ &= (e(x) \bullet p' + e(x) \bullet p + e(y) \bullet q' + e(y) \bullet q + p' \bullet q' + p' \bullet q + p \bullet q', x \bullet y). \end{aligned}$$

Since the two sides are equal, we know that their first components must be equal, so we have

$$\partial(q) \blacktriangleright p' + \partial(p) \blacktriangleright q' = q \bullet p' + p \bullet q'$$

and

$$\begin{aligned} q \bullet p' + p \bullet q' &= \partial(q) \blacktriangleright p' + \partial(p) \blacktriangleright q' \\ &= \partial(q + p) \blacktriangleright (p' + q') - \partial(q) \blacktriangleright q' - \partial(p) \blacktriangleright p' \\ &= \partial(q + p) \blacktriangleright (p' + q') - (q \bullet q' + p \bullet p'), \end{aligned}$$

thus

$$\begin{aligned}\partial(q+p) \blacktriangleright (p'+q') &= q \bullet p' + p \bullet q' + (q \bullet q' + p \bullet p') \\ &= (q+p) \bullet (q'+p')\end{aligned}$$

and writing  $(q+p) = l, (q'+p') = l' \in Kers$ , we get

$$\partial(l) \blacktriangleright l' = l \bullet l'$$

which is the Peiffer identity as required. Hence  $(Kers, A_0, \partial)$  is a crossed module.

Let  $\mathcal{A} = (A_0, A_1, s, t, e, \circ, \bullet)$  and  $\mathcal{A}' = (A'_0, A'_1, s', t', e', \circ', \bullet')$  be 2-algebras and  $F = (F_0, F_1) : \mathcal{A} \rightarrow \mathcal{A}'$  be a 2-algebra morphism. Then  $F_0 : A_0 \rightarrow A'_0$  and  $F_1 : A_1 \rightarrow A'_1$  are the  $k$ -algebra morphisms. We define  $f_1 = F_1|_{Kers} : Kers \rightarrow Kers'$  and  $f_0 = F_0 : A_0 \rightarrow A'_0$ . For all  $a \in Kers$  and  $x \in A_0$ ,

$$\begin{aligned}f_0\partial(a) &= F_0t(a) \\ &= t'F_1(a) \\ &= \partial'f_1(a)\end{aligned}$$

and

$$\begin{aligned}f_1(x \blacktriangleright a) &= F_1(e(x)a) \\ &= F_1(e(x))F_1(a) \\ &= e'F_0(x)F_1(a) \\ &= e'f_0(x)f_1(a) \\ &= f_0(x) \blacktriangleright f_1(a).\end{aligned}$$

Thus  $(f_1, f_0)$  map is a crossed module morphism  $(Kers, A_0, \partial) \rightarrow (Kers', A'_0, \partial')$ . So we have a functor

$$\Gamma : \mathbf{2Alg} \rightarrow \mathbf{XMod}_k.$$

Conversely, let  $(G, C, \partial)$  be a crossed module of algebras. Therefore there is an algebra morphism  $\partial : G \rightarrow C$  and an action of  $C$  on  $G$  such that

$$\text{CM1) } \partial(x \blacktriangleright g) = x\partial(g),$$

$$\text{CM2) } \partial(g) \blacktriangleright g' = gg'.$$

Since  $C$  acts on  $G$ , we can form the semidirect product  $G \rtimes C$  as defined by

$$G \rtimes C = \{(g, c) \mid g \in G, c \in C\}$$

with multiplication

$$(g, c) (g', c') = (c \blacktriangleright g' + c' \blacktriangleright g + g'g, cc')$$

and define maps  $s, t : G \rtimes C \rightarrow C$  and  $e : C \rightarrow G \rtimes C$  by  $s(g, c) = c$ ,  $t(g, c) = \partial(g) + c$  and  $e(c) = (0, c)$ . These maps are clearly algebra morphisms.

$$\begin{array}{ccc}
& c & \\
& \curvearrowright & \\
* & \Downarrow (g,c) & * \\
& \partial(g)+c & \\
& \curvearrowleft & \\
& \Downarrow (g',\partial(g)+c) & \\
& \partial(g+g')+c & 
\end{array}$$

For  $t(g, c) = s(g', \partial(g) + c) = \partial(g) + c$ , we define composition

$$\begin{aligned}
\circ : (G \rtimes C)_t \times_s (G \rtimes C) &\longrightarrow (G \rtimes C) \\
((g, c), (g', \partial(g) + c)) &\longmapsto (g + g', c),
\end{aligned}$$

for  $(g, c), (h, d) \in G \rtimes C$  and  $(g, c), (g', \partial(g) + c) \in G \rtimes C$ , following equations give horizontal and vertical composition respectively.

$$(g, c) \bullet (h, d) = (c \blacktriangleright h + d \blacktriangleright g + gh, cd)$$

$$(g, c) \circ (g', \partial(g) + c) = (g + g', c)$$

Finally, since it must be that  $\circ$  is an algebra morphism and by the crossed module conditions, interchange law is satisfied. Therefore we have constructed a 2-algebra  $\mathcal{A} = (C, G \rtimes C, s, t, e, \circ, \bullet)$  consists of the single object say  $*$  and the  $k$ -algebra  $C$  of 1-morphisms and the  $k$ -algebra  $G \rtimes C$  of 2-morphisms. Let  $(G, C, \partial)$  and  $(G', C', \partial')$  be crossed modules and  $f = (f_1, f_0) : (G, C, \partial) \longrightarrow (G', C', \partial')$  be a crossed module morphism. We define

$$\begin{aligned}
F_1 : G \rtimes C &\longrightarrow G' \rtimes C' \\
(g, c) &\longmapsto F_1(g, c) = (f_1(g), f_0(c))
\end{aligned}$$

and

$$\begin{aligned}
F_0 : C &\longrightarrow C' \\
c &\longmapsto F_0(c) = f_0(c).
\end{aligned}$$

Then

$$\begin{aligned}
s' F_1(g, c) &= s'(f_1(g), f_0(c)) \\
&= f_0(c) \\
&= F_0(c) \\
&= F_0 s(g, c),
\end{aligned}$$

$$\begin{aligned}
t' F_1(g, c) &= t'(f_1(g), f_0(c)) \\
&= \partial' f_1(g) + f_0(c) \\
&= f_0 \partial(g) + f_0(c) \\
&= F_0(\partial(g) + c) \\
&= F_0 t(g, c),
\end{aligned}$$

$$\begin{aligned}
e' F_0(c) &= (0, f_0(c)) \\
&= F_1(0, c) \\
&= F_1 e(c),
\end{aligned}$$

$$\begin{aligned}
F_1((g, c) \circ (g', c')) &= F_1(g + g', c) \\
&= (f_1(g + g'), f_0(c)) \\
&= (f_1(g) + f_1(g'), f_0(c)) \\
&= (f_1(g), f_0(c)) \circ (f_1(g'), f_0(c')) \\
&= F_1(g, c) \circ F_1(g', c'),
\end{aligned}$$

$$\begin{aligned}
F_1((g, c) \bullet (h, d)) &= F_1(c \blacktriangleright h + d \blacktriangleright g + gh, cd) \\
&= (f_1(c \blacktriangleright h) + f_1(d \blacktriangleright g) + f_1(gh), f_0(cd)) \\
&= (f_0(c) \blacktriangleright f_1(h) + f_0(d) \blacktriangleright f_1(g) + f_1(g)f_1(h), f_0(c)f_0(d)) \\
&= (f_1(g), f_0(c)) \bullet (f_1(h), f_0(d)) \\
&= F_1(g, c) \bullet F_1(h, d)
\end{aligned}$$

for all  $(g, c) \in G \rtimes C$  and  $c \in C$ . Therefore  $F = (F_1, F_0)$  is a 2-algebra morphism from  $(C, G \rtimes C, s, t, e, \circ, \bullet)$  to  $(C', G' \rtimes C', s', t', e', \circ', \bullet')$ . Thus we get a functor

$$\Psi : \mathbf{XMod}_k \longrightarrow \mathbf{2Alg}.$$

□

### 3.1 Homotopies of Crossed modules and 2-algebras

The notion of homotopy for morphisms of crossed modules over commutative algebras is given in [1]. In this section, we explain the relation between homotopies for crossed modules over commutative algebras and homotopies for 2-algebras. The formulae given below are playing important role in our study.

**Definition 3.2** [1] *Let  $\mathcal{A} = (E, R, \partial)$  and  $\mathcal{A}' = (E', R', \partial')$  be crossed modules and  $f_0 : R \longrightarrow R'$  be an algebra morphism. An  $f_0$ -derivation  $s : R \longrightarrow E'$  is a  $k$ -linear map satisfying for all  $r, r' \in R$ ,*

$$s(rr') = f_0(r) \blacktriangleright s(r') + f_0(r') \blacktriangleright s(r) + s(r)s(r').$$

*Let  $f = (f_1, f_0)$  be a crossed module morphism  $\mathcal{A} \longrightarrow \mathcal{A}'$  and  $s$  be an  $f_0$ -derivation. If  $g = (g_1, g_2)$  is defined as (where  $e \in E$  and  $r \in R$ )*

$$\begin{aligned}
g_0(r) &= f_0(r) + (\partial' s)(r) \\
g_1(e) &= f_1(e) + (s\partial)(e),
\end{aligned}$$

*then  $g$  is also crossed module morphism  $\mathcal{A} \longrightarrow \mathcal{A}'$ . In such a case we write  $f \xrightarrow{(f_0, s)} g$ , and say that  $(f_0, s)$  is a homotopy connecting  $f$  to  $g$ .*

If  $(f_0, s)$  and  $(g_0, s')$  are homotopies connecting  $f$  to  $g$  and  $g$  to  $u$  respectively, then  $(f_0, s + s')$  is a homotopy connecting  $f$  to  $u$ , where  $s + s' : R \longrightarrow E'$  is an  $f_0$ -derivation defined by  $(s + s')(r) = s(r) + s'(r)$ .

The notion of homotopy for 2-algebras is essentially a special case of 2-natural transformation due to Gray in [11].

**Definition 3.3** Let  $\mathbf{A} = (A_0, A_1, s, t, e, \circ, \bullet)$  and  $\mathbf{A}' = (A'_0, A'_1, s', t', e', \circ', \bullet')$  be 2-algebras and let  $F = (F_1, F_0)$  and  $G = (G_1, G_0)$  be 2-algebra morphisms  $\mathbf{A} \rightarrow \mathbf{A}'$ . A  $k$ -algebra morphism  $\delta : A_0 \rightarrow A'_1$  satisfying the following conditions is called a homotopy connecting  $F$  to  $G$  :

- 1)  $s' \delta = F_0$
- 2)  $t' \delta = G_0$
- 3)  $F_1 \circ' \delta t = \delta s \circ' G_1$ . In such a case we write  $F \xrightarrow{\delta} G$ .

**Theorem 3.4** Let  $\mathcal{A} = (A_0, A_1, s, t, e, \circ, \bullet)$ ,  $\mathcal{A}' = (A'_0, A'_1, s', t', e', \circ', \bullet')$  be 2-algebras,  $F = (F_1, F_0)$ ,  $G = (G_1, G_0)$  and  $U = (U_1, U_0)$  be 2-algebra morphisms  $\mathcal{A} \rightarrow \mathcal{A}'$  and  $\delta$  be a homotopy connecting  $F$  to  $G$ ,  $\delta'$  be a homotopy connecting  $G$  to  $U$ . Then the map  $\delta * \delta' : A_0 \rightarrow A'_1$  defined by  $(\delta * \delta')(x) = \delta(x) + \delta'(x) - e'(t' \delta)(x)$  is a homotopy connecting  $F$  to  $U$ .

**Proof:** We first show that  $\delta * \delta'$  is an algebra morphism. Since  $\delta$  and  $\delta'$  are algebra morphisms,  $\delta(x \bullet x') = \delta(x) \bullet \delta(x')$  and  $\delta'(x \bullet x') = \delta'(x) \bullet \delta'(x')$  for all  $x, x' \in A_0$ . Then we get

$$\begin{aligned}
(\delta * \delta')(x \bullet x') &= \delta(x \bullet x') + \delta'(x \bullet x') - e'(t' \delta)(x \bullet x') \\
&= \delta(x \bullet x') + \delta'(x \bullet x') - e'(G_0)(x \bullet x') \\
&= \delta(x \bullet x') \circ' \delta'(x \bullet x') \quad (\text{Proposition 2.2}) \\
&= (\delta(x) \bullet \delta(x')) \circ' (\delta'(x) \bullet \delta'(x')) \\
&= (\delta(x) \circ' \delta'(x)) \bullet (\delta(x') \circ' \delta'(x')) \quad (\text{interchange law}) \\
&= (\delta(x) + \delta'(x) - e'(G_0)(x)) \bullet (\delta(x') + \delta'(x') - e'(G_0)(x')) \\
&= (\delta * \delta')(x) \bullet (\delta * \delta')(x').
\end{aligned}$$

For all  $x \in A_0$

$$\begin{aligned}
s'(\delta * \delta')(x) &= s'(\delta(x) + \delta'(x) - e' G_0(x)) \\
&= s' \delta(x) + s' \delta'(x) - s' e' G_0(x) \\
&= F_0(x) + G_0(x) - G_0(x) \\
&= F_0(x),
\end{aligned}$$

$$\begin{aligned}
t'(\delta * \delta')(x) &= t'(\delta(x) + \delta'(x) - e' G_0(x)) \\
&= t' \delta(x) + t' \delta'(x) - t' e' G_0(x) \\
&= G_0(x) + U_0(x) - G_0(x) \\
&= U_0(x),
\end{aligned}$$

and since  $F_1 \circ' \delta t = \delta s \circ' G_1$  and  $G_1 \circ' \delta' t = \delta' s \circ' U_1$ , we get

$$\begin{aligned}
F_1 \circ' \delta t \circ' \delta' t &= \delta s \circ' G_1 \circ' \delta' t \\
&= \delta s \circ' \delta' s \circ' U_1.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
F_1 \circ' (\delta * \delta') t &= F_1 \circ' (\delta t \circ' \delta' t) \\
&= (\delta s \circ' \delta' s) \circ' U_1 \\
&= (\delta * \delta') s \circ' U_1.
\end{aligned}$$

Therefore  $\delta * \delta' : A_0 \longrightarrow A_1$  is a homotopy connecting  $F$  to  $U$ .  $\square$

**Theorem 3.5** *Let  $\Gamma : 2\text{Alg} \longrightarrow X\text{Mod}_k$  be the functor as mentioned in Theorem 3.1 and  $\delta$  be homotopy connecting  $F$  to  $G$ . Then*

$$\begin{aligned} \Gamma(\delta) = h & : A_0 \longrightarrow Kers' \\ x & \longmapsto h(x) = \delta(x) - e'(s'\delta)(x) \end{aligned}$$

*is a homotopy of corresponding crossed module morphisms.*

**Proof:** We first show that  $h$  is an  $f_0$ -derivation where  $f_0 : A_0 \longrightarrow A'_0$  defined by  $f_0(x) = F_0(x)$ . For  $x, x' \in A_0$ ,

$$\begin{aligned} f_0(x) \blacktriangleright h(x') & \\ + f_0(x') \blacktriangleright h(x) + h(x) \bullet' h(x') & = F_0(x) \blacktriangleright (\delta(x') - e'(s'\delta)(x')) \\ & \quad + F_0(x') \blacktriangleright (\delta(x) - e'(s'\delta)(x)) \\ & \quad + (\delta(x) - e'(s'\delta)(x)) \bullet' (\delta(x') - e'(s'\delta)(x')) \\ & = e'(F_0(x)) \bullet' (\delta(x') - e'F_0(x')) \\ & \quad + e'(F_0(x')) \bullet' (\delta(x) - e'F_0(x)) + \delta(x) \bullet' \delta(x') \\ & \quad - \delta(x) \bullet' e'F_0(x') - e'F_0(x) \bullet' \delta(x') + e'F_0(x) \bullet' e'F_0(x') \\ & = \delta(x \bullet x') - e'(s'\delta)(x \bullet x') \\ & = h(x \bullet x'). \end{aligned}$$

Therefore  $h$  is an  $f_0$ -derivation.

Now we show that

$$\begin{aligned} g_0(x) & = f_0(x) + \partial' h(x) \\ g_1(n) & = f_1(n) + h\partial(n) \end{aligned}$$

for  $x \in A_0$  and  $n \in Kers$ .

$$\begin{aligned} \partial' h(x) & = \partial'(\delta(x) - e'f_0(x)) \\ & = \partial'(\delta(x)) - \partial'(e'f_0(x)) \\ & = (t'\delta)(x) - (t'e')f_0(x) \\ & = g_0(x) - f_0(x) \end{aligned}$$

and we get  $g_0(x) = f_0(x) + \partial' h(x)$ .

Since  $A_1 \simeq Kers \rtimes A_0$ , we take  $a = (n, x)$  for  $a \in A_1$  where  $n = a - es(a) \in Kers$  and  $x = s(a) \in A_0$ . We define  $\delta^* : A_0 \longrightarrow Kers' \rtimes A'_0$ , as  $\delta^*(x) = (\delta(x) - e's'(\delta(x)), s'\delta(x))$  and  $h^* : A_0 \longrightarrow Kers' \rtimes A'_0$ , as  $h^*(x) =$

$(h(x), F_0(x))$ . Therefore

$$\begin{array}{ccc}
 A_1 \cong \text{Ker}(s) \rtimes A_0 & \begin{array}{c} \xleftarrow{e} \\ \xrightarrow{s} \\ \xrightarrow{t} \end{array} & A_0 \\
 \downarrow (G_1, G_0) & \searrow (F_1, F_0) \quad \delta^* & \downarrow G_0 \\
 A'_1 \cong \text{Ker}(s') \rtimes A'_0 & \begin{array}{c} \xleftarrow{e'} \\ \xrightarrow{s'} \\ \xrightarrow{t'} \end{array} & A'_0 \\
 & & \downarrow F_0
 \end{array}$$

for  $(F_1, F_0)(n, x), (\delta^* t)(n, x) \in A_1 \cong \text{Ker}(s') \rtimes A'_0$  such that  $t(F_1, F_0)(n, x) = s(\delta^* t)(n, x)$ , we have  $(F_1, F_0)(n, x) \circ' \delta^* t(n, x) = (F_1(n) + \delta t(n), F_0(x))$  and  $-(F_1, F_0)(n, x) = (-F_1(n), t' F_1(n) + F_0(x))$  and then, since

$$(F_1, F_0)(n, x) \circ' \delta^* t(n, x) = \delta^* s(n, x) \circ' (G_1, G_0)(n, x)$$

we have

$$\begin{aligned}
 \delta^* t(n, x) &= -(F_1, F_0)(n, x) \circ' \delta^* s(n, x) \circ' (G_1, G_0)(n, x) \\
 &= (-F_1(n) + h(x) + G_1(n), t' F_1(n) + F_0(x))
 \end{aligned}$$

and

$$-e' F_0 t(n, x) = (0, t' f_1(n) + f_0(x)).$$

Hence we get

$$\begin{aligned}
 \delta^* t(n, x) - e' F_0 t(n, x) &= (I_{t' F_1(n) + F_0(x)} \circ \delta t)(n, x) \\
 &= \delta^* t(n, x).
 \end{aligned}$$

Then

$$\begin{aligned}
 h^*(t(n, x)) &= \delta^*(t(n, x)) - e'(s' \delta^*)(t(n, x)) \\
 &= \delta^* t(n, x) - e' F_0 t^*(n, x) \\
 &= \delta^* t(n, x) \\
 &= (-F_1(n) + h(x) + G_1(n), t' F_1(n) + F_0(x))
 \end{aligned} \tag{1}$$

and

$$\begin{aligned}
 h^*(t(n, x)) &= h^*(\partial(n) + x) \\
 &= (h(\partial(n) + x), f_0(\partial(n) + x)) \\
 &= (h(\partial(n)) + h(x), f_0(\partial(n)) + f_0(x)) \\
 &= (h(\partial(n)) + h(x), t' F_1(n) + F_0(x)).
 \end{aligned} \tag{2}$$

Therefore from (1) and (2) we have

$$h(\partial(n)) + h(x) = -F_1(n) + h(x) + G_1(n)$$

and

$$h(\partial(n)) = -F_1(n) + G_1(n).$$

Then

$$g_1(n) = f_1(n) + h\partial(n).$$

Hence

$$\begin{aligned} h : A_0 &\longrightarrow Kers' \\ x &\longmapsto h(x) = \delta(x) - e' F_0(x) \end{aligned}$$

is a homotopy connecting  $f = (f_1, f_0) : (Kers \xrightarrow{\partial} A_0) \longrightarrow (Kers' \xrightarrow{\partial'} A'_0)$  to  $g = (g_1, g_0) : (Kers \xrightarrow{\partial} A_0) \longrightarrow (Kers' \xrightarrow{\partial'} A'_0)$ .

Let  $F \xrightarrow{\delta} G$  and  $G \xrightarrow{\delta'} H$ . Then we have

$$\begin{aligned} \Gamma(\delta * \delta')(x) &= (\delta * \delta')(x) - e'(s' \delta * \delta')(x) \\ &= \delta(x) + \delta'(x) - e'(t' \delta)(x) - e'(s' \delta)(x) \\ &= \delta(x) + \delta'(x) - e'(s' \delta')(x) - e'(s' \delta)(x) \\ &= (\delta(x) - e'(s' \delta)(x)) + (\delta'(x) - e'(s' \delta')(x)) \\ &= \Gamma(\delta)(x) + \Gamma(\delta')(x) \end{aligned}$$

for all  $x \in A_0$ .  $\square$

**Theorem 3.6** *Let  $\Psi : XMod_k \longrightarrow 2Alg$  be the functor as mentioned in Theorem 3.1 and  $h$  be homotopy connecting  $f : (G, C, \partial) \longrightarrow (G', C', \partial')$  to  $g : (G, C, \partial) \longrightarrow (G', C', \partial')$ . Then*

$$\begin{aligned} \Psi(h) = \delta : C &\longrightarrow G' \rtimes C' \\ x &\longmapsto \delta(x) = (h(x), f_0(x)) \end{aligned}$$

*is a homotopy of corresponding 2-algebra morphisms.*

**Proof:** We first show that  $\delta$  is an algebra morphism. For  $x, x' \in C$

$$\begin{aligned} \delta(xx') &= (h(xx'), f_0(xx')) \\ &= (f_0(x) \blacktriangleright h(x') + f_0(x') \blacktriangleright h(x) + h(x)h(x'), f_0(x)f_0(x')) \\ &= (h(x), f_0(x))(h(x'), f_0(x')) \\ &= \delta(x)\delta(x'). \end{aligned}$$

Now we show that

$$\begin{aligned} 1) s' \delta &= F_0 & 2) t' \delta &= G_0 & 3) (f_1, f_0) \circ' \delta t &= \delta s \circ' (g_1, g_0) \\ 1) \text{For all } x \in C, & & & & \end{aligned}$$

$$\begin{aligned} s' \delta(x) &= s'(h(x), f_0(x)) \\ &= f_0(x) = F_0(x), \end{aligned}$$



2) For all  $x \in C$ ,

$$\begin{aligned} t' \delta(x) &= t'(h(x), f_0(x)) \\ &= t'(h(x)) + f_0(x) \\ &= \partial' h(x) + f_0(x) \\ &= g_0(x) = G_0(x), \end{aligned}$$

3) For all  $x \in C, a \in G$ , since  $t'(f_1(a), f_0(x)) = \partial' f_1(a) + f_0(x)$ ,

$$\begin{aligned} s'(\delta t(a, x)) &= s'(\delta(\partial(a) + x)) \\ &= s'(h(\partial(a) + x), f_0(\partial(a) + x)) \\ &= f_0(\partial(a) + x) \\ &= f_0(\partial(a)) + f_0(x) \\ &= \partial' f_1(a) + f_0(x) \end{aligned}$$

then  $t'(f_1(a), f_0(x)) = s'(\delta t(a, x))$  and  $(f_1, f_0)$ ,  $\delta t$  are composable pairs.

Also since

$$\begin{aligned} t'(\delta s(a, x)) &= t'(\delta(x)) = t'(h(x), f_0(x)) \\ &= \partial'(h(x)) + f_0(x) \\ &= g_0(x) \end{aligned}$$

and  $s'(g_1(a), g_0(x)) = g_0(x)$  then  $t'(\delta s) = s'(g_1, g_0)$  and  $\delta s, (g_1, g_0)$  are composable pairs.

Therefore we get

$$(f_1(a), f_0(x)) \circ' \delta t(a, x) = (f_1(a) + h(\partial(a) + x), f_0(x))$$

and

$$\delta s(a, x) \circ' (g_1(a), g_0(x)) = (f_1(a) + h(\partial(a) + x), f_0(x)).$$

Then  $(f_1, f_0) \circ' \delta t = \delta s \circ' (g_1, g_0)$ . So

$$\begin{aligned} \delta: C &\longrightarrow G' \rtimes C' \\ c &\longmapsto \delta(x) = (h(x), f_0(x)) \end{aligned}$$

is a homotopy connecting  $F = ((f_1, f_0), f_0)$  to  $G = ((g_1, g_0), g_0)$ .

Let  $f \xrightarrow{h} g$  and  $g \xrightarrow{h'} u$ . Then we have

$$\begin{aligned} \Psi(h + h')(x) &= ((h + h')(x), f_0(x)) \\ &= (h(x) + h'(x), f_0(x)) \\ &= (h(x), f_0(x)) + (h'(x), g_0(x)) - (0, g_0(x)) \\ &= \Psi(h)(x) + \Psi(h')(x) - e'(t'(\Psi)(h))(x) \\ &= (\Psi(h) * \Psi(h))(x). \end{aligned}$$

□

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